

The ergodic problem for some subelliptic operators with unbounded coefficients

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Abstract

We study existence and uniqueness of the invariant measure for a stochastic process with degenerate diffusion, whose infinitesimal generator is a linear subelliptic operator in the whole space \mathbb{R}^N with coefficients that may be unbounded. Such a measure together with a Liouville-type theorem will play a crucial role in two applications: the ergodic problem studied through stationary problems with vanishing discount and the long time behavior of the solution to a parabolic Cauchy problem. In both cases, the constants will be characterized in terms of the invariant measure.

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1 Introduction

This paper is devoted to study with pde's methods, the existence and uniqueness of the invariant measure of stochastic processes with degenerate diffusion, whose infinitesimal generators are linear subelliptic operators in the whole space \mathbb{R}^N with coefficients that may be unbounded. The invariant measures play a crucial role in ergodicity, homogenization and large time behaviour of the value function associated to the process. These

methods, based on optimal control theory and pde's arguments, were introduced in the 80's by Bensoussan and developed until nowadays (see the monograph [8] by Bensoussan and references therein).

We shall first tackle the case of the Heisenberg group as model problem; after we shall extend our techniques to other subelliptic operators. In the Heisenberg case, we consider the stochastic dynamics

$$(1.1) \quad dX_t = b(X_t)dt + \sqrt{2}\sigma(X_t)dW_t \quad \text{for } t \in (0, +\infty), \quad X_0 = x^0 \in \mathbb{R}^3$$

where, if $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, the matrix $\sigma(x)$ has the form

$$(1.2) \quad \sigma(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2x_2 & -2x_1 \end{bmatrix}$$

(in other words, the columns of σ are vectors generating the Heisenberg group) while W_t is a 3-dimensional Brownian motion.

Our principal aim is to prove, under suitable assumptions on the drift b , the existence and uniqueness of the invariant measure m associated to the process (1.1).

Let us recall from [8] that a probability measure m on \mathbb{R}^3 is an *invariant measure* for process (1.1) if, for each $u_0 \in \mathcal{L}^\infty(\mathbb{R}^3)$, it satisfies

$$(1.3) \quad \int_{\mathbb{R}^3} u(x, t) m(x) dx = \int_{\mathbb{R}^3} u_0(x) m(x) dx$$

where $u(x, t) = \mathbb{E}_x(u_0(X_t))$ is the solution to the parabolic Cauchy problem

$$\begin{cases} \partial_t u + \mathcal{L}u = 0 & \text{in } (0, +\infty) \times \mathbb{R}^3 \\ u(0, x) = u_0(x) & \text{on } \mathbb{R}^3 \end{cases}$$

and

$$(1.4) \quad -\mathcal{L}u := \text{tr}(\sigma(x)\sigma^T(x)D^2u(x)) + b(x) \cdot Du(x)$$

is the *infinitesimal generator* of process (1.1).

It is well known (see [8, Sect. II.4 and II.5]) that the density of the probability m (which, with a slight abuse of notation, we still denote by m) solves

$$\mathcal{L}^*m = 0, \quad \int_{\mathbb{R}^3} m dx = 1 \quad \text{and} \quad m \geq 0,$$

where \mathcal{L}^*m is the adjoint operator

$$\mathcal{L}^*m = - \sum_{i,j} \partial_{ij}((\sigma\sigma^T)_{ij}m) + \sum_i \partial_i(b_i m).$$

In the framework of locally strongly elliptic operators, Has'minskiĭ [18, Sect. IV.4] (see also [27, Sect.8.2]) established the existence of an invariant measure provided that there exists a bounded set U with smooth boundary such that

$$(1.5) \quad \begin{cases} \text{for any } x^0 \in \mathbb{R}^N \setminus U, \text{ the mean time } \tau \text{ at which the path (1.1) issuing} \\ \text{from } x^0 \text{ reaches } U \text{ is finite and } \mathbb{E}_x \tau \text{ is locally finite.} \end{cases}$$

In our case this result does not apply because the matrix $A := \sigma\sigma^T$ with σ given by (1.2) is

$$(1.6) \quad A(x) = \begin{bmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \\ 2x_2 & -2x_1 & 4(x_1^2 + x_2^2) \end{bmatrix}$$

and it is only positive semidefinite.

It is worth noticing (see [5, 15, 26]) that a sufficient condition for property (1.5) is the existence of a *Lyapunov*-like function w which satisfies, for some positive constants k and R_0

$$(1.7) \quad \mathcal{L}w \geq k \quad \text{for } |x| \geq R_0 \quad \text{and} \quad w(x) \rightarrow +\infty \quad \text{as } |x| \rightarrow +\infty.$$

As one can easily check the presence of the first order term is somehow 'crucial' for the existence of such a function. We will prove the existence of such Lyapunov function under suitable assumptions on the drift b that include also the Ornstein-Uhlenbeck case (see [27] and Remark 2.3 below) where the operator is of the following type

$$-\mathcal{L}u := \text{tr}(\sigma(x)\sigma^T(x)D^2u(x)) - \alpha x \cdot Du(x), \quad \alpha > 0.$$

For ergodicity results based on probabilistic methods we refer to [21] and [24] and the references therein. The existence of a Lyapunov function is reminiscent of similar conditions (for instance, see: [27, Sect. 8.2] and [30, 31, 32]) called "recurrence condition" in the probabilistic jargon.

Ichihara and Kunita [20] (see also [23]) proved the existence of an invariant measure for hypoelliptic processes as (1.1) which are constrained

in a compact set. It is worth to recall that, in unbounded set the existence of an invariant measure may fail as it can be easily seen for (1.1) with $b = 0$ and $\sigma = I$.

In this paper we want to establish existence and uniqueness of an invariant measure for process (1.1), namely for a process with the following features: it lies in an unbounded set and its infinitesimal generator is simultaneously degenerate and with unbounded coefficients. To this end we shall use only pure analytical arguments.

It is important to stress that, in the Heisenberg case, the principal part of $\mathcal{L}u$ can be written as $\sum_{i=1}^2 X_i^2 u$ where X_1, X_2 , are the vector fields given by the columns of σ and that they satisfy Hörmander condition: X_1, X_2 , and their commutators of any order span \mathbb{R}^3 at each point $(x_1, x_2, x_3) \in \mathbb{R}^3$. In this case we have that $[X_1, X_2] = -4\partial_{x_3}$. This property will play a crucial role in this paper since, as for the uniformly elliptic case, we have regularity, comparison and maximum principle ([13]).

The methods used in this work are strongly inspired by the lectures "Equations paraboliques et ergodicité" of P.L Lions at Collège de France (2014-15) [25] and by a unpublished manuscript by P.L. Lions and M. Musiela [26] (see also the paper of Cirant [15] for similar arguments). Actually, we shall consider the process

$$(1.8) \quad dX_t^\rho = b(X_t^\rho)dt + \sqrt{2}\sigma_\rho(X_t^\rho)dW_t,$$

where σ_ρ is the approximating matrix of σ in (1.2):

$$\sigma_\rho(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x_2 & -2x_1 & \rho \end{bmatrix}$$

such that $A_\rho = \sigma_\rho \sigma_\rho^T$ is locally strictly positive, constrained in a bounded set O_n suitably chosen.

Let us stress that, in our argument, it is not enough to approximate the matrix A with any non-degenerate matrix A_ρ but we also need that A_ρ can be written as $\sigma_\rho \sigma_\rho^T$, where σ_ρ is the diffusion matrix of a new underlying optimal control problem. This issue motivates the fact that in (1.8) a new Brownian motion appears.

Let us recall from [8] that the invariant measure m_ρ^n of this process solves

$$\mathcal{L}_\rho^* m_\rho^n = 0 \quad \text{in } O_n$$

coupled with a boundary condition of Neumann type, where

$$-\mathcal{L}_\rho(u) = \text{tr}(\sigma_\rho(x)\sigma_\rho^T(x)D^2u(x)) + b(x) \cdot Du(x)$$

is an uniformly elliptic operator in O_n . Letting $n \rightarrow +\infty$, we obtain an invariant measure m_ρ for the process (1.8) in the whole space; letting $\rho \rightarrow 0^+$, we get the desired invariant measure for (1.1). The Lyapunov function will play a crucial role in these limits: it will be used in order to prove that all the m_ρ 's and m are really measures (in other words, that the m_ρ^n and the m_ρ do not “disperse at infinity”).

Moreover in this paper we also establish a Liouville type result. Similar result for semilinear operator without the drift term can be founded in the papers [9, 10, 14] and references therein; in all these papers the nonlinear zeroth order term is the key ingredient whereas, in our setting, the crucial contribution is due to the drift.

We shall use the invariant measure and the Liouville property in two classic applications: an ergodic problem and the long time behaviour of a Cauchy problem. For the former problem we consider the family of equations

$$(1.9) \quad \delta u_\delta - \text{tr}(\sigma(x)\sigma^T(x)D^2u_\delta) - b(x)Du_\delta = F(x) \quad \text{in } \mathbb{R}^3,$$

where $\delta > 0$ and we shall prove that, as $\delta \rightarrow 0$, δu_δ converges to a constant λ , called “ergodic” constant. Let us stress that the differential operator in the ergodic problem coincides with the infinitesimal generator \mathcal{L} of process (1.1).

We recall that the study of ergodic problems for equations with periodic, uniformly elliptic, operators has been addressed in [3, 7] while, for periodic, possibly degenerate (still satisfying the Hörmander’s condition) operators, we refer the reader to the papers [1, 2].

The main difficulties in our problem are the lack of periodicity and the degeneracy of the operator. We shall overcome these issues using some techniques introduced by [5] for an elliptic operator on the whole space. Moreover, we shall give an explicit formula for the ergodic constant λ in terms of the invariant measure for (1.1).

In the latter application we consider the following Cauchy problem:

$$u_t + \mathcal{L}u = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^3, \quad u(0, x) = f(x) \quad \text{on } \mathbb{R}^3,$$

where \mathcal{L} is the operator defined in (1.4). We will prove that, as $t \rightarrow +\infty$, the solution u converges to a constant Λ which will be characterised in terms of the invariant measure.

Finally, we shall show how to extend our previous results to other degenerate operators satisfying Hörmander condition with possibly unbounded coefficients.

Our future purpose is to use the ergodic problem to study the homogenization problem

$$(1.10) \quad -\epsilon \operatorname{tr}(\sigma(\frac{x}{\epsilon})\sigma^T(\frac{x}{\epsilon})D^2u_\epsilon) - b(\frac{x}{\epsilon}) \cdot Du_\epsilon + f(x, \frac{x}{\epsilon}) + au_\epsilon = 0 \text{ in } \mathbb{R}^3,$$

where σ has the form (1.2). In this case the approximated cell problem formally coincides with the problem (1.9).

For the study of homogenization problems for periodic, possibly nonlinear, degenerate (still satisfying the Hörmander's condition) operators, we refer the reader to the papers [1, 11, 28].

This paper is organized as follows: Section 2 contains the main result of the paper: we find conditions on the drift b such that a Lyapunov functions does exist and by means of this function we prove the existence and uniqueness of an invariant measure associated to our process. In Section 3, we establish a Liouville type result assuming the existence of a Lyapunov-like function. Section 4 is devoted to our applications: in Section 4.1 we study the ergodic problem through stationary problems with vanishing discount, while in Section 4.2 we consider the long time behaviour of a Cauchy problem. In Section 5 we generalise the previous results to a more general class of subelliptic operators, encompassing *e.g.* the Grushin one. The Appendix contains a condition equivalent to (1.7) which will be useful to manage the Lyapunov function founded in Section 2.

2 Existence and uniqueness of the invariant measure

This section is devoted to the invariant measure for process (1.1). Let us recall (see [26] or Proposition 2.1 below) that, when the matrix associated to the infinitesimal generator \mathcal{L} is a strictly definite positive matrix, a sufficient condition for the existence of an invariant measure is given by: there exists a *Lyapunov-like* function such that

$$(2.1) \quad \begin{aligned} w &\in C^\infty(B_0^C) \cap C^0(\mathbb{R}^3) \\ \mathcal{L}w &\geq 1, \text{ in } B_0^C \\ w &\geq 0 \text{ in } B_0^C, \quad w = 0 \text{ on } \partial B_0, \end{aligned}$$

where B_0 is a ball centered in 0 with suitable radius. (For less regular functions w , we refer to ([26])).

In our case, the matrix $A = \sigma\sigma^T$ in (1.6) is degenerate in any point, and the rank of the matrix is 2. In order to overcome this issue, for $\rho > 0$, we introduce the approximating operators

$$(2.2) \quad \mathcal{L}_\rho w := -\text{tr}(A_\rho(x)D^2w) - b(x)Dw,$$

where

$$(2.3) \quad A_\rho(x) = \begin{bmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \\ 2x_2 & -2x_1 & 4(x_1^2 + x_2^2) + \rho^2 \end{bmatrix} = \sigma(x)\sigma^T(x) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho^2 \end{bmatrix}.$$

In the following Lemma we collect some useful properties of \mathcal{L}_ρ .

Lemma 2.1 *The matrix $A_\rho(x)$ is locally strictly positive definite (namely, for any compact $K \subset \mathbb{R}^3$, there holds $\lambda A_\rho(x)\lambda^T \geq \nu(x)|\lambda|^2$ for any $x \in K$, with $\nu(x) \geq a(K, \rho) > 0$) and it is positive definite in \mathbb{R}^3 .*

Moreover, there exists a 3×3 matrix $\sigma_\rho(x)$ with linear coefficients such that

$$(2.4) \quad A_\rho(x) = \sigma_\rho(x)\sigma_\rho^T(x).$$

Proof. Set $\alpha = 4(x_1^2 + x_2^2) + \rho^2 + 1$. The eigenvalues of A_ρ are

$$\lambda_1 = 1, \quad \lambda_{2,3} = \frac{\alpha \pm \sqrt{\alpha^2 - 4\rho^2}}{2}.$$

It is easy to remark that $\lambda_2 \geq \frac{1}{2}$.

The last eigenvalue is $\lambda_3 = \frac{2\rho^2}{\alpha + \sqrt{\alpha^2 - 4\rho^2}} > \frac{\rho^2}{\alpha}$ hence, for any fixed $R > 0$, if

$x_1^2 + x_2^2 \leq R^2$, $\alpha \leq 4R^2 + \rho^2 + 1$ and $\lambda_3 > \frac{\rho^2}{4R^2 + \rho^2 + 1} > 0$.

The matrix

$$(2.5) \quad \sigma_\rho(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2x_2 & -2x_1 & \rho \end{bmatrix}.$$

verifies (2.4)

□

Remark 2.1 From (2.4), beside being uniformly elliptic, the operator $-\mathcal{L}_\rho$ is also the infinitesimal generator of the stochastic process

$$(2.6) \quad dX_t^\rho = b(X_t^\rho)dt + \sqrt{2}\sigma_\rho(X_t^\rho)dW_t,$$

where σ_ρ is defined in (2.5) and $W_t = (W_{1t}, W_{2t}, W_{3t})$ and W_{1t}, W_{2t}, W_{3t} are three independent Brownian motions whereas our starting process (1.1) only contains two independent Brownian motions.

Now, we want to prove that, for some classes of drifts b , there exists a function w satisfying (2.1) with \mathcal{L} replaced by \mathcal{L}_ρ . To this end, we consider a continuous drift $b = (b_1, b_2, b_3)$ such that

$$(2.7) \quad b_i(x) = b_i(x_i), \quad \begin{cases} b_i(x_i) \leq -\frac{C_i}{|x_i|^{1-\alpha}} & \text{for } x_i \geq R \\ b_i(x_i) \geq \frac{C_i}{|x_i|^{1-\alpha}} & \text{for } x_i \leq -R \end{cases}$$

for some constants $\alpha \geq 0$, $R > 0$ and $C_i > 0$ ($i = 1, 2, 3$).

Note that Lemma 2.2 here below, holds also for $\rho = 0$ then we have a Lyapunov-like function w (i.e. satisfying condition (2.1)) also for the degenerate starting problem where \mathcal{L} is given by (3.1).

Similar conditions to (2.7) was obtained in [26] with $\sigma = I$ the identity matrix.

Lemma 2.2 *Assume σ as in (1.2). Assume that b is a continuous function verifying (2.7) with*

- (i) *either $\alpha > 0$,*
- (ii) *or $\alpha = 0$ and sufficiently large C_i .*

Then, there exists a R_0 and a C^∞ function w which satisfies

$$(2.8) \quad \mathcal{L}_\rho w \geq 1 \quad \text{in } \overline{B(0, R_0)}^C, \quad w \geq 0 \quad \text{in } \overline{B(0, R_0)}^C, \quad \lim_{|x| \rightarrow \infty} w = \infty$$

for ρ sufficiently small.

Proof. We set

$$w := \frac{(x_1^4 + x_2^4)}{12} + \frac{x_3^2}{2}.$$

Then, there holds

$$\mathcal{L}_\rho w = -5(x_1^2 + x_2^2) - \rho^2 - \frac{1}{3}(b_1 x_1^3 + b_2 x_2^3) - b_3 x_3.$$

We denote $K_i := \max_{x_i \in [-R, R]} |b_i(x_i)|$.

Case (i). Assume $\alpha > 0$. We want to prove that there exists R_0 such that $\mathcal{L}_\rho w > 1$ in $\overline{B(0, R_0)}^C$ for ρ sufficiently small. To this end, we split the arguments in several cases.

(I). If $|x_i| \geq R$ for any $i \in \{1, 2, 3\}$, then

$$\mathcal{L}_\rho w \geq x_1^2(-5 + C_1|x_1|^\alpha/3) + x_2^2(-5 + C_2|x_2|^\alpha/3) + C_3|x_3|^\alpha - \rho^2.$$

Hence, for $|x_1|, |x_2| > R_1 := \max\{(15/C_1)^{1/\alpha}, (15/C_2)^{1/\alpha}, R\}$, $|x_3| \geq R_3 := \max\{C_3^{-1/\alpha}, R\}$, we get: $\mathcal{L}_\rho w \geq 1$ for ρ sufficiently small.

(II). If $|x_1|, |x_2| \leq R_1$ and $|x_3| \geq R$, then

$$\mathcal{L}_\rho w \geq -10R_1^2 - \rho^2 - R^3(K_1 + K_2)/3 + C_3|x_3|^\alpha$$

(here, we used the relation: $-b_i x_i^3 \geq 0$ for $|x_i| \in [R, R_1]$, $i = 1, 2$). Hence, for $|x_3| \geq \tilde{R}_3$ with \tilde{R}_3 sufficiently large, taking ρ sufficiently small, we get $\mathcal{L}_\rho w \geq 1$.

(III). If $|x_1| \leq R_1$, $|x_2| \geq R_1$ and $|x_3| \geq R$ (and similarly, for $|x_1| \geq R_1$, $|x_2| \leq R_1$ and $|x_3| \geq R$), then

$$\mathcal{L}_\rho w \geq -5R_1^2 + x_2^2(-5 + |x_2|^\alpha/3) - \rho^2 - K_1 R^3/3 + C_3|x_3|^\alpha.$$

Hence, for $|x_3| \geq \tilde{R}_3$, we get $\mathcal{L}_\rho w \geq 1$ for ρ sufficiently small.

(IV). If $|x_1| \leq R_1$, $|x_2| > R$, $|x_3| < \tilde{R}_3$ (and similarly for $|x_1| > R$, $|x_2| \leq R_1$, $|x_3| < \tilde{R}_3$), then

$$\mathcal{L}_\rho w \geq |x_2|^2(-5 + C_2|x_2|^\alpha/3) - 5R_1^2 - \rho^2 - K_1 R^2/3 - K_3 R.$$

Hence, for $|x_2| > R_1$, we get $\mathcal{L}_\rho w \geq 1$ for ρ sufficiently small.

In conclusion, gluing together all these cases, we accomplish the proof for $\alpha > 0$.

Case (ii). Assume $\alpha = 0$; we want to prove that there exist some constants C_i and a radius R_0 such that $\mathcal{L}_\rho w \geq 1$ in $\overline{B(0, R_0)}^C$.

(I). If $|x_i| > R$ for any $i = 1, 2, 3$, then $\mathcal{L}_\rho w \geq x_1^2(-5 + \frac{1}{3}C_1) + x_2^2(-5 + \frac{1}{3}C_1) + C_3 - \rho^2$; hence, for $C_1, C_2 > 15$, $C_3 > 1$, we have $\mathcal{L}_\rho w > 1$ for ρ sufficiently small.

(II). If $|x_i| < R$ for $i = 1, 2$ and $|x_3| > R$, then $\mathcal{L}_\rho w \geq -10R^2 - R^3(K_1 + K_2)/3 - \rho^2 + C_3$; hence, for $C_3 > 10R^2 + R^3(K_1 + K_2)/3 + 1$, we have $\mathcal{L}_\rho w > 1$ for ρ sufficiently small.

(III). If $|x_1| < R$, $|x_2| > R$ and $|x_3| > R$ (and similarly, for $|x_1| \geq R$, $|x_2| \leq R$ and $|x_3| \geq R$), then $\mathcal{L}_\rho w \geq -5R^2 + x_2^2(-5 + C_2/3) - R^2 K_1/3 - \rho^2 + C_3$,

hence, for $C_2 > 15$, C_3 sufficiently large and ρ sufficiently small, we have $\mathcal{L}_\rho w > 1$.

(IV). If $|x_1| \leq R$, $|x_2| > R$, $|x_3| < R$ (and similarly for $|x_1| > R$, $|x_2| \leq R$, $|x_3| < R$), then $\mathcal{L}_\rho w \geq -5R^2 + x_2^2(-5 + C_2/3) - K_1 R^2 - \rho^2 - K_3 R$; hence, for $C_2 > 15$, $|x_2|$ sufficiently large and ρ sufficiently small, we have $\mathcal{L}_\rho w > 1$. \square

Remark 2.2 Stronger sufficient condition on b_i for the existence of a Lyapunov-like function w satisfying condition (2.1) could be found using $w(x) := \log((x_1^2 + x_2^2)^2 + x_3^2)$.

Remark 2.3 The drifts of the Ornstein-Uhlenbeck operator (i.e., $b(x) = -\gamma x$ for $\gamma > 0$) satisfy assumption (i) of Lemma 2.2. For further properties of this operator we refer the reader to the monograph [27].

In the next proposition we will establish the existence of an invariant measure m_ρ of the approximating process (2.6). This measure will be used in the main theorem of this paper when the invariant measure for the process (1.1) will be obtained as the limit of m_ρ as $\rho \rightarrow 0$.

Proposition 2.1 *Let $\sigma_\rho(x)$ defined by (2.5) and $b(x)$ be a Lipschitz function satisfying (2.7) either with $\alpha > 0$ or $\alpha = 0$ and C_i sufficiently large. There exists a unique invariant probability measure m_ρ on \mathbb{R}^3 for the process (2.6).*

Proof. As proved in Lemma 2.1 the operator \mathcal{L}_ρ is uniformly elliptic in each bounded set (but the ellipticity constant degenerates in the whole \mathbb{R}^3). We adapt some techniques introduced by [26] (see also [15] for similar arguments), by considering approximate problems in domains O_n such that $O_n \nearrow \mathbb{R}^3$ if $n \rightarrow +\infty$.

Let us recall (see [25] or Lemma A.1 in the Appendix) that condition (2.8) is equivalent to the following one:

$$(2.9) \quad \begin{aligned} &\text{there exists a function } \overline{w} \in C^\infty(\mathbb{R}^3) \text{ such that} \\ &\mathcal{L}_\rho \overline{w} + \chi \overline{w} = \phi \quad \text{in } \mathbb{R}^3, \quad \lim_{|x| \rightarrow +\infty} \overline{w} = \infty \end{aligned}$$

where $\chi \in C_0^\infty$ and $\phi \in C^\infty$ are suitable functions such that, $\chi > 0$ on B_0 , $\text{supp} \chi = \bar{B}_0$ (B_0 is a suitable open set) and $\lim_{|x| \rightarrow \infty} \phi = \infty$. (As a matter of facts, this condition is satisfied by the function w chosen in the proof

of Lemma 2.2-(i)).

We define $O_n := \{x \in \mathbb{R}^3 \mid \bar{w}(x) < M_n\}$ where $M_n \rightarrow +\infty$ if $n \rightarrow +\infty$ and M_n is not a critical value of \bar{w} . Since $\bar{w} \rightarrow +\infty$ if $x \rightarrow +\infty$ then O_n are bounded and smooth and $O_n \nearrow \mathbb{R}^3$.

Fix $\rho > 0$ and n , the results by Bensoussan [8, Section 4] ensure that there exists an unique invariant measure m_ρ^n associated to the diffusion process X_t^ρ in O_n with reflecting boundary whose infinitesimal generator is L_ρ in O_n with boundary conditions

$$\sum_{i,j} (a_\rho)_{ij} \frac{\partial u}{\partial \nu_j} = 0 \quad \text{on } \partial O_n$$

where ν denotes the unit outward normal to ∂O_n and the matrix $A_\rho = (a_\rho)_{ij} = \sigma_\rho \sigma_\rho^T$ as in Lemma 2.1.

The invariant measure m_ρ^n satisfies the problem

$$(2.10) \quad \mathcal{L}_\rho^* m_\rho^n := - \sum_{i,j} \frac{\partial^2 ((a_\rho)_{ij} m_\rho^n)}{\partial x_i \partial x_j} + \sum_i \frac{\partial (b_i m_\rho^n)}{\partial x_i} = 0 \quad \text{in } O_n,$$

$$(2.11) \quad \sum_{ij} \nu_i \left(\frac{\partial ((a_\rho)_{ij} m_\rho^n)}{\partial x_j} - b_i m_\rho^n \right) = 0 \quad \text{on } \partial O_n$$

$$\int_{O_n} m_\rho^n = 1, \quad m_\rho^n > 0.$$

We have to prove that, as $n \rightarrow +\infty$, m_ρ^n converges in some sense to m_ρ invariant measure to the process with generator \mathcal{L}_ρ , i.e. m_ρ solves

$$(2.12) \quad \mathcal{L}_\rho^* m_\rho := - \sum_{i,j} \frac{\partial^2 ((a_\rho)_{ij} m_\rho)}{\partial x_i \partial x_j} + \sum_i \frac{\partial (b_i m_\rho)}{\partial x_i} = 0 \quad \text{in } \mathbb{R}^3$$

$$\int_{\mathbb{R}^3} m_\rho = 1, \quad m_\rho \geq 0.$$

From Prohorov Theorem and the fact that $\int_{O_n} m_\rho^n = 1$ we know that $m_\rho^n \rightharpoonup m_\rho$ as $n \rightarrow +\infty$ (possibly passing to a subsequence).

We prove now that $\int_{\mathbb{R}^3} m_\rho = 1$. Multiplying equation (2.10) by \bar{w} defined in (2.9), integrating on O^n and taking into account (2.11) we obtain

$$0 = \int_{O^n} \mathcal{L}_\rho^* m_\rho^n \bar{w} = \int_{O^n} m_\rho^n \mathcal{L}_\rho \bar{w} + \int_{\partial O^n} m_\rho^n \sum_{i,j} (a_\rho)_{ij} \frac{\partial \bar{w}}{\partial x_i} \nu_j.$$

Since $\bar{w} = M_n$ on ∂O^n and $\bar{w} < M_n$ on O^n , we have $\frac{\partial \bar{w}}{\partial x_i} = \frac{\partial w}{\partial \nu} \nu_i$ and $\frac{\partial \bar{w}}{\partial \nu} \geq 0$ on ∂O^n . Then, there holds

$$0 = \int_{O^n} m_\rho^n \mathcal{L}_\rho \bar{w} + \frac{\partial \bar{w}}{\partial \nu} \int_{\partial O^n} m_\rho^n \sum_{i,j} (a_\rho)_{ij} \nu_i \nu_j,$$

and since $\sum_{i,j} (a_\rho)_{ij} \nu_i \nu_j \geq 0$, we obtain $\int_{O^n} m_\rho^n \mathcal{L}_\rho \bar{w} \leq 0$. Hence

$$\int_{O^n} m_\rho^n \mathcal{L}_\rho \bar{w} = \int_{O^n} (\phi - \chi \bar{w}) m_\rho^n \leq 0,$$

and

$$\int_{O^n} \phi m_\rho^n \leq \int_{\text{supp} \chi} \chi \bar{w} m_\rho^n \leq C$$

where C is a positive constant independent of n . Let us extend m_ρ^n by zero outside O^n , and call it again m_ρ^n , then

$$(2.13) \quad \int_{\mathbf{R}^3} \phi m_\rho^n \leq C$$

where C is a positive constant independent of n .

Since $\lim_{|x| \rightarrow +\infty} \phi(x) = +\infty$, for any N there exists a R_N such that $\phi(x) > N$ on $B_{R_N}^C$. Hence, from (2.13)

$$(2.14) \quad \int_{B_{R_N}^C} m_\rho^n \leq \frac{C}{N}.$$

Since $\int_{\mathbf{R}^3} m_\rho^n = 1$ then from (2.14)

$$\int_{B_{R_N}} m_\rho^n \geq 1 - \frac{C}{N}$$

and from the weak convergence of m_ρ^n to m_ρ , we have

$$\int_{B_{R_N}} m_\rho \geq 1 - \frac{C}{N},$$

hence letting $N \rightarrow +\infty$ we obtain that $\int_{\mathbf{R}^3} m_\rho = 1$.

Moreover from the local regularity $W^{2,p}$ for any $p > 1$ of m_ρ^n since m_ρ^n

solves equation (2.10), passing to the limit we easily obtain that m_ρ solves equation (2.12). \square

Remark 2.4 The condition of strict ellipticity in the compact subsets of \mathbb{R}^N is sufficient to deduce from (2.8) the existence of the invariant measure m_ρ (see [18, Theorem IV.4.1] under their assumption B.1). Nevertheless we gave the proof of Proposition 2.1 because it is purely analytic and for the sake of completeness.

Now we want to prove that m_ρ converges in some sense to m , invariant measure to the process (1.1), solving

$$(2.15) \quad \mathcal{L}^* m = 0, \quad \int_{\mathbb{R}^3} m = 1 \quad \text{and} \quad m \geq 0.$$

Theorem 2.1 *Let σ be defined by (1.2) and $b(y)$ be a continuous function satisfying (2.7) either with $\alpha > 0$ or $\alpha = 0$ and C_i sufficiently large. Then there exists a unique invariant probability measure m on \mathbb{R}^3 for the process (1.1).*

Proof. The uniqueness of the measure m comes from the results of Arnold, Klieman [4], or Ichihara, Kunita [20].

The existence of the invariant measure it is obtained proving that the invariant measure m_ρ of Proposition 2.1 converges, if ρ tends to 0, to the measure m associated to the process (1.1). We proceed analogously to Proposition 2.1. The measure m_ρ satisfies the following conditions:

$$(2.16) \quad \mathcal{L}_\rho^* m_\rho = 0 \quad \text{in } \mathbb{R}^3, \quad \int_{\mathbb{R}^N} m_\rho = 1, \quad m_\rho \geq 0.$$

We know that $m_\rho \rightarrow m$ as $\rho \rightarrow 0$ (at least for a subsequence) where m is a measure. We have to prove that m is an invariant measure to the process (1.1) i.e. that m solves (2.15).

From condition (2.8) and the equivalent conditions (2.9), we know that there exists smooth functions χ and ϕ such that \overline{w} satisfies $\mathcal{L}_\rho \overline{w} + \chi \overline{w} = \phi$, in \mathbb{R}^3 , \overline{w} and ϕ such that $\rightarrow +\infty$ if $|x| \rightarrow +\infty$ and χ has compact support.

Multiplying equation (2.12) by such \overline{w} and integrating on \mathbb{R}^3 we obtain

$$0 = \int_{\mathbb{R}^3} \mathcal{L}_\rho^* m_\rho \overline{w} = \int_{\mathbb{R}^3} \mathcal{L}_\rho \overline{w} m_\rho = \int_{\mathbb{R}^3} (\phi - \chi \overline{w}) m_\rho,$$

hence

$$(2.17) \quad \int_{\mathbb{R}^3} \phi m_\rho = \int_{\text{supp } \chi} \chi \bar{w} m_\rho \leq C,$$

where C is a positive constant independent of ρ . From (2.17), since

$$1 = \int_{B_{R_N}} m_\rho + \int_{B_{R_N}^C} m_\rho,$$

then

$$\int_{B_{R_N}} m_\rho \geq 1 - \frac{C}{N}$$

and from the convergence of m_ρ

$$\int_{B_{R_N}} m \geq 1 - \frac{C}{N},$$

hence letting $N \rightarrow +\infty$ we obtain $\int_{\mathbb{R}^3} m = 1$.

To prove that $\mathcal{L}^* m = 0$ we write, for any ψ smooth,

$$0 = \int_{\mathbb{R}^3} \mathcal{L}_\rho^* m_\rho \psi = \int_{\mathbb{R}^3} \mathcal{L}_\rho \psi m_\rho \rightarrow \int_{\mathbb{R}^3} \mathcal{L} \psi m = \int_{\mathbb{R}^3} \mathcal{L}^* m \psi.$$

Taking account that $\mathcal{L}_\rho \psi \rightarrow \mathcal{L} \psi$ strongly and $m_\rho \rightharpoonup m$ weakly in L^1 . \square

3 A Liouville type result

In this section, we establish a Liouville type result, which holds true not only in the Heisenberg setting but also for σ whose columns satisfy the general Hörmander condition. This result will be stated in Proposition 3.1. Although in the proof of Theorems 4.1 and 4.2 below it will be applied to the particular case of a regular solution, Proposition 3.1 contains a general statement which has its own independent interest.

Let us first recall from [19] the definition of Hörmander condition.

Definition 3.1 *The vector fields X_j , $j = 1, \dots, m$, satisfy the Hörmander condition if X_1, \dots, X_m and their commutators of any order span \mathbb{R}^N at each point of \mathbb{R}^N .*

Proposition 3.1 *Consider the problem*

$$(3.1) \quad \mathcal{L}V = -\text{tr}(\sigma(x)\sigma^T(x)D^2V) - b(x) \cdot DV = 0, \quad x \in \mathbb{R}^N$$

where σ and b are smooth functions and the vector fields $X_j = \sigma^j \cdot \nabla$, $j = 1, \dots, m$ satisfy the Hörmander condition as in definition (3.1). Assume that there exists $w(x) \in C^\infty(\mathbb{R}^N)$ and $R_0 > 0$ such that

$$(3.2) \quad \mathcal{L}w \geq 0 \quad \text{in } \overline{B(0, R_0)}^C, \quad w(x) \rightarrow +\infty \quad \text{as } |x| \rightarrow +\infty.$$

Then:

- (i) every viscosity subsolution $V \in USC(\mathbb{R}^N)$ to (3.1) such that $\limsup_{|x| \rightarrow +\infty} \frac{V}{w} \leq 0$ is constant;
- (ii) every viscosity supersolution $V \in LSC(\mathbb{R}^N)$ to (3.1) such that $\liminf_{|x| \rightarrow +\infty} \frac{V}{w} \geq 0$ is constant.

Proof. The proof uses the same arguments as in [26] (see also [5, Lemma 4.1 and remark 4.1]). For the sake of completeness, we shall give the proof of case (i); being similar, the proof of case (ii) is omitted.

Let us first observe that if $\psi \in \mathcal{C}^2(A)$ (A is any open set $A \subset \overline{B(0, R_0)}^C$) is a classical supersolution in A , i.e.

$$\mathcal{L}\psi \geq 0 \text{ in } A$$

then $w + \psi$ is a viscosity supersolution in A , i.e.

$$\mathcal{L}(w + \psi) \geq 0 \text{ in } A.$$

Define for each $\eta > 0$:

$$V_\eta := V(x) - \eta w(x).$$

We claim that V_η is a viscosity subsolution in $\overline{B(0, R_0)}^C$ i.e.

$$(3.3) \quad \mathcal{L}(V_\eta) \leq 0 \text{ in } \overline{B(0, R_0)}^C.$$

Indeed, let us assume by contradiction that there exists $\psi \in \mathcal{C}^2(\overline{B(0, R_0)}^C)$ such that $V_\eta - \psi$ attains a strict maximum in some point $\bar{x} \in \overline{B(0, R_0)}^C$, $V(\bar{x}) = \eta w(\bar{x}) + \psi(\bar{x})$, and that there holds

$$\mathcal{L}(\psi)(\bar{x}) > 0.$$

By the continuity of the coefficients of L , and the regularity of ψ there exists a $r_0 > 0$ such that

$$(3.4) \quad \mathcal{L}(\psi)(x) > 0 \text{ in } B(\bar{x}, r_0) \subset \overline{B(0, R_0)}^C.$$

As remarked above $\eta w + \psi$ is a supersolution in $B(\bar{x}, r_0)$. Moreover there exists $\alpha > 0$ such that $V(x) < \eta w(x) + \psi(x) - \alpha$ for any $x \in \partial B(\bar{x}, r_0)$. Then by a local comparison principle (see [6] or [13] for classical solutions), $V(x) \leq \eta w(x) + \psi(x) - \alpha$ in $B(\bar{x}, r_0)$ and for $x = \bar{x}$ we get a contradiction and our claim (3.3) is proved.

Thanks to $V_\eta \rightarrow -\infty$ as $|x| \rightarrow +\infty$, there exist $R_1(\eta) = R_1 > R_0$ such that

$$V_\eta(x) \leq \sup_{|z|=R_0} V_\eta(z), \quad \forall |x| \geq R_1$$

then, using the weak maximum principle applied to V_η ,

$$\max_{B(0, R_1) \setminus \overline{B(0, R_0)}} V_\eta = \max_{\partial B(0, R_0)} V_\eta$$

and this implies that

$$V_\eta(x) \leq \max_{\partial B(0, R_0)} V_\eta, \quad \forall x \in \overline{B(0, R_0)}^C.$$

Letting $\eta \rightarrow 0$ in the preceding inequality:

$$V(x) \leq \max_{\partial B(0, R_0)} V, \quad \forall x \in \overline{B(0, R_0)}^C.$$

Therefore V attains its global maximum so it is a constant by the strong maximum principle established by Bardi and Da Lio [6, Corollary 3.2]. \square

Remark 3.1 Note that, in the Heisenberg group, the function w introduced in Lemma 2.2 satisfies assumptions (3.2).

Remark 3.2 Let us stress that the above arguments work for any linear operator \mathcal{L} with continuous coefficients, satisfying a local comparison principle and a strong maximum principle.

Remark 3.3 Note that conditions on the sub and super solutions in i) and ii) imply the boundedness of the sub and super solutions.

4 Applications

In this section we provide two applications of the previous results. In both cases we will use the existence of the invariant measure for the process (1.1) proved in Section 2 and the Liouville type property obtained in Section 3. Summarizing, we shall prove that

$$\lim_{\delta \rightarrow 0} \delta u_\delta(x) = \lim_{t \rightarrow +\infty} u(t, x) = \lim_{t \rightarrow +\infty} \frac{v(t, x)}{t} = \int_{\mathbb{R}^3} f(x) dm(x),$$

where m is the invariant measure of Section 2 and u_δ , u and w are the solutions respectively of

$$\begin{aligned} \delta u_\delta(x) + \mathcal{L}u &= f(x), & \text{in } \mathbb{R}^3, \\ u_t + \mathcal{L}u &= 0, \quad u(0, x) = f(x), & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ v_t + \mathcal{L}v &= f, \quad v(0, x) = 0, & \text{in } (0, +\infty) \times \mathbb{R}^3, \end{aligned}$$

and \mathcal{L} is the infinitesimal generator of the process (1.1), i.e. the operator defined in (1.4).

4.1 The ergodic problem

In this section we tackle the following ergodic problem. We consider the family of problems

$$(4.1) \quad \delta u_\delta(x) - \text{tr}(\sigma(x)\sigma^T(x)D^2u_\delta) - b(x)Du_\delta = f(x) \quad \text{in } \mathbb{R}^3,$$

where $\delta > 0$ and we investigate about the convergence as $\delta \rightarrow 0$ of δu_δ to a constant λ called the ergodic constant. Throughout this section, we assume

(A₁) σ is defined in (1.2)

(A₂) $b \in C^\infty(\mathbb{R}^3)$ and satisfies the hypotheses of Lemma 2.2

(A₃) $f \in C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$

The next two Lemma contain several properties of u_δ which will be used later on.

Lemma 4.1 *Under Assumptions (A₁)-(A₃), there exists an unique smooth viscosity solution u_δ of the approximating problem (4.1) such that*

$$(4.2) \quad |u_\delta(x)| \leq \frac{C}{\delta}, \quad \forall x \in \mathbb{R}^3,$$

for some positive constant C independent of δ .

Proof. The uniqueness follows from the comparison principle proved in [13]. By assumption (A_3) it is easy to see that $w^\pm = \pm \frac{C}{\delta}$ with C sufficiently large is respectively a supersolution and a subsolution for problem (4.1). In conclusion, applying Perron's method, we infer the existence of a solution to (4.1) verifying (4.2). \square

Lemma 4.2 *Under assumptions (A_1) - (A_3) , the functions $v_\delta := \delta u_\delta$, where u_δ is the solution of problem (4.1), are locally uniformly Hölder continuous. Namely, there exists $\alpha \in (0, 1)$ such that for every compact $K \subset \mathbb{R}^3$ there exists a constant N such that*

$$(4.3) \quad |v_\delta(x_1) - v_\delta(x_2)| \leq N|x_1 - x_2|^\alpha, \quad \forall x_1, x_2 \in K, \quad \forall \delta \in (0, 1).$$

The constant N only depends on K and on the data of the problem (in particular is independent of δ).

Proof. The statement is a direct consequence of the result of Krylov [22]. For the sake of completeness let us sketch how to apply Krylov's result to our case. From Lemma 4.1 the function v_δ is uniformly bounded and smooth and solves the following equation

$$(4.4) \quad \delta v_\delta - \text{tr}(\sigma(x)\sigma^T(x)D^2v_\delta) - b(x)Dv_\delta = \delta f(x) \quad \text{in } \mathbb{R}^3.$$

We observe that equation (4.4) can be written in the form

$$(4.5) \quad -L_0 v_\delta + v_\delta := -\sigma^{ik}\partial_{x_i}(\sigma^{jk}\partial_{x_j}v_\delta) - BDv_\delta + v_\delta = \delta f + (1 - \delta)v_\delta$$

where $B_j = b_j - \sum_{jk} \sigma^{ik}\partial_{x_i}\sigma^{jk}$ and $L_0 = \sigma^{ik}\partial_{x_i}(\sigma^{jk}\partial_{x_j}\cdot)$.

For δ fixed, consider the problem

$$\begin{cases} -L_0 v_{\delta,n} + v_{\delta,n} = \delta f + (1 - \delta)v_\delta & \text{in } B(0, n) \\ v_{\delta,n} = 0 & \text{on } \partial B(0, n). \end{cases}$$

We observe that $\{v_{\delta,n}\}$ is a equibounded family (by the same arguments of Lemma 4.1). [22, Theorem 2.1] of Krylov ensures that there exists $\alpha \in (0, 1)$ such that for every compact $K \subset \mathbb{R}^3$ there exists a constant N_1 (independent of δ, n) such that

$$(4.6) \quad |v_{\delta,n}(x_1) - v_{\delta,n}(x_2)| \leq N_1|x_1 - x_2|^\alpha, \quad \forall x_1, x_2 \in K.$$

By Ascoli-Arzelà Theorem, letting $n \rightarrow +\infty$ (possibly passing to a subsequence) we get that $v_{\delta,n}$ converges locally uniformly to a function V_δ . By the stability and uniqueness results we infer $V_\delta = v_\delta$. Moreover, passing to the limit in n in (4.6), we get (4.3). \square

In the next result we prove that δu_δ converges to a constant which will be characterize in terms of the invariant measure of the process (1.1).

Theorem 4.1 *Under assumptions (A_1) - (A_3) , the solution u_δ of problem (4.1) given in Lemma 4.1 satisfies*

$$(4.7) \quad \lim_{\delta \rightarrow 0} \delta u_\delta = \int_{\mathbb{R}^3} f(x) dm(x), \text{ locally uniformly,}$$

where m is the invariant measure of process (1.1) founded in Section 2.

Proof. We shall proceed following some arguments of [5]. The functions $v_\delta := \delta u_\delta$ solve (4.4) and, from estimate (4.2), satisfy

$$(4.8) \quad |v_\delta| \leq C, \quad \text{in } \mathbb{R}^3,$$

with C independent of δ , hence they are uniformly bounded in \mathbb{R}^3 . From Lemma 4.2 v_δ are also uniformly Hölder continuous in any compact set of \mathbb{R}^3 . Then by the Ascoli-Arzelà theorem there is a sequence $\delta_n \rightarrow 0$ and a continuous function w such that $v_{\delta_n} \rightarrow w$ locally uniformly; by stability, w is a solution of

$$(4.9) \quad -tr(\sigma(x)\sigma^T(x)D^2v) - b(x)Dv = 0, \quad x \in \mathbb{R}^3,$$

hence $w \in C^\infty$ by the hypoellipticity of the operator (see [13]). Then by Proposition 3.1, w is constant.

In conclusion, we have that, possibly passing to a subsequence, $\{\delta u_\delta\}_\delta$ converges locally uniformly to a constant. Now, it remains to prove that this constant is independent of the subsequence chosen and that it has the form (4.7). By standard arguments of optimal control theory (see [17]), the function u_δ can be written as

$$u_\delta(x) = \mathbb{E}_x \int_0^{+\infty} f(X_t) e^{-\delta t} dt$$

where X_t is the process in (1.1) with initial data $X_0 = x$ while \mathbb{E} denotes the expectation. Integrating both sides with respect to the invariant measure, we infer

$$\begin{aligned} \int_{\mathbb{R}^3} u_\delta(x) dm(x) &= \int_0^{+\infty} \left(\mathbb{E}_x \int_{\mathbb{R}^3} f(X_t) dm(x) \right) e^{-\delta t} dt \\ &= \int_0^{+\infty} \left(\int_{\mathbb{R}^3} f(x) dm(x) \right) e^{-\delta t} dt \\ &= \frac{1}{\delta} \int_{\mathbb{R}^3} f(x) dm(x) \end{aligned}$$

where the second inequality is due to the definition of invariant measure. Taking into account that every convergent subsequence of $\{\delta u_\delta\}_\delta$ must converge to a constant, we conclude that all the sequence $\{\delta u_\delta\}_\delta$ converges to $\int_{\mathbb{R}^3} f \, dm$. \square

4.2 Large time behavior of solutions

This section concerns the asymptotic behavior for large times of the solution of the parabolic Cauchy problem:

$$(4.10) \quad \begin{cases} u_t + \mathcal{L}u = 0 & \text{in } (0, +\infty) \times \mathbb{R}^3, \\ u(0, x) = f(x) & \text{on } \mathbb{R}^3, \end{cases}$$

where \mathcal{L} is the operator defined in (1.4). Let us recall that, for periodic fully nonlinear equations, this issue was studied in [1, Theorem 4.2]. We quote here also the results in the manuscript [26].

Theorem 4.2 *Under the assumptions of Theorem 2.1 and Proposition 3.1, for $f(x) \in C^0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, the solution $u(t, x)$ of problem (4.10) verifies*

$$\lim_{t \rightarrow +\infty} u(t, x) = \int_{\mathbb{R}^3} f(x) \, dm(x), \quad \text{locally uniformly in } x,$$

where m is the invariant measure of process (1.1) founded in Section 2.

Proof. Since $\pm \|f\|_\infty$ are sub and supersolution of (4.10), by the comparison principle we have that

$$(4.11) \quad \|u\|_\infty \leq \|f\|_\infty.$$

Arguing as in [1, Theorem 4.2] we get, for some $c > 0$, $|u(t+s, x) - u(t, x)| \leq cs$, and in particular $|u_t(t, x)| \leq c$. Moreover classical results on regularity of subelliptic operators give that $u(t, \cdot)$ are locally Hölder continuous on x uniformly in t (see [13, 22]). Hence by Ascoli-Arzelà theorem for any sequence $t_n \rightarrow +\infty$ there exists a subsequence t_{n_k} such that $u(t_{n_k}, \cdot) \rightarrow v$ locally uniformly for some $v \in C^0(\mathbb{R}^3)$. By standard arguments (see [1, Theorem 4.2]), v is the solution of $\mathcal{L}v = 0$; hence, by Proposition 3.1 (the Liouville type result), it is a constant. Therefore, we have

$$(4.12) \quad u(t_{n_k}, \cdot) \rightarrow C, \quad \text{locally uniformly}.$$

We show now that the constant C is independent of the chosen sequence. Let us consider an arbitrary sequence s_n such that $s_n \rightarrow +\infty$ and $u(s_n, \cdot) \rightarrow K$ locally uniformly. From (1.3)

$$\int_{\mathbb{R}^3} u(s_n, x) dm(x) = \int_{\mathbb{R}^3} f(x) dm(x)$$

Using (4.11), $\int_{\mathbb{R}^3} dm(x) = 1$ and the dominated convergence theorem

$$K = \int_{\mathbb{R}^3} f(x) dm(x).$$

□

Remark 4.1 Let us consider the following Cauchy problems

$$(4.13) \quad \begin{cases} v_t + \mathcal{L}v = f & \text{in } (0, +\infty) \times \mathbb{R}^3 \\ v(0, x) = 0 & \text{on } \mathbb{R}^3, \end{cases}$$

where \mathcal{L} is the operator defined in (1.4) and f is a function as in Theorem 4.2.

By means of the Duhamel formula and a change of variables the solution v can be written as $v(t, x) = \int_0^t u(\tau, x) d\tau$ where u is the solution of (4.10). Hence the statement of Theorem 4.2 can be rephrased as

$$\lim_{t \rightarrow +\infty} \frac{v(t, x)}{t} = \int_{\mathbb{R}^3} f(x) dm(x).$$

5 The general case

In this section we address to the process (1.1) under the following assumptions:

$$(5.1) \quad \begin{cases} \sigma(x) \in C^\infty(\mathbb{R}^N), \\ \|\sigma(x)\| \leq C(|x| + 1), & \text{for some } C > 0; \\ \text{the columns of } \sigma \text{ satisfy Hörmander condition.} \end{cases}$$

$$(5.2) \quad b(x), f(x) \in C^\infty(\mathbb{R}^N), \quad \|b(x)\|, |f(x)| \leq C(|x| + 1), \quad C > 0.$$

$$(5.3) \quad \begin{cases} \text{For } A := \sigma\sigma^T, \text{ there exists } \{A_\rho(x)\}_{\rho \in (0,1)} \text{ with } A_\rho = \sigma_\rho\sigma_\rho^T, \\ \sigma_\rho \in C^\infty(\mathbb{R}^N), \quad A_\rho \rightarrow A \text{ in } L^\infty \text{ and } A_\rho \text{ is locally definite positive.} \end{cases}$$

(5.4)

There exists a function w which verifies (2.8) for any ρ sufficiently small.

The grow assumptions on σ in (5.1) and on b and f in (5.2) allow us to obtain the existence of a process X_t in (1.1).

Under assumptions (5.1), (5.2), the Liouville type result contained in Proposition 3.1 still holds true. In fact the results of Bony [13] on comparison principle and strong maximum principle hold also in this setting if we observe that

$$-tr(\sigma\sigma^T D^2u) = \sum_j X_j^2 u - C(x) \cdot Du,$$

where $C(x) = D\sigma^j \cdot \sigma^j$ and σ^j are the columns of the matrix σ .

Theorem 5.1 *Under assumptions (5.1)-(5.4) there exists an invariant probability measure m associated to the diffusion process (1.1).*

Proof. We observe that, by assumptions (5.3), (5.4), there exists an unique invariant measure m_ρ for the process with diffusion σ_ρ . Then, arguing as in the proof of Theorem 2.1, using again the function w in (5.4) we obtain the existence of the invariant measure associated to the process (1.1). \square

Corollary 5.1 *Under assumptions (5.1)-(5.4) Theorems 4.1 and 4.2 hold true.*

Example 5.1 *The Grushin operator*

For $x = (x_1, x_2) \in \mathbb{R}^2$, consider the diffusion matrix

$$(5.5) \quad \sigma(x) = \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix}$$

and observe that σ satisfies (5.1) since $X_1 = (1, 0)$ and $[X_1, X_2] = (0, 1)$ span all \mathbb{R}^2 . In this case the infinitesimal generator is

$$\mathcal{L}V = -V_{x_1 x_1} - x_1^2 V_{x_2 x_2} - b(x)DV.$$

We take f and $b(x) = (b_1(x_1), b_2(x_2))$ satisfying (5.2) with

$$(5.6) \quad \begin{cases} -b_1 x_1 \geq 6, & \text{if } |x_1| > 1, \\ -b_2 x_2 \geq 1, & \text{if } |x_2| > 1, \\ b_i \text{ are bounded in } [-1, 1], & i = 1, 2. \end{cases}$$

Under these assumptions it is easy to check that the matrix

$$\sigma_\rho(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_1 & \rho \end{pmatrix}$$

satisfies (5.3) and that the function $W(x) = \frac{1}{12}x_1^4 + \frac{1}{2}x_2^2$ satisfies (5.4). In conclusion, since all the hypotheses (5.1)-(5.4) are satisfied, Theorem 5.1 apply.

Remark 5.1 Lions-Musiela in [26] have considered a similar degenerate case but in their paper the elements of the matrix are bounded in \mathbb{R}^2 in this way

$$(5.7) \quad \sigma(x) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{x_1}{\sqrt{1+x_1^2}} \end{pmatrix}.$$

A Appendix

In the following Lemma we state the equivalence between conditions (2.8) and (2.9). This property has already been established by P.L. Lions [25]; however, for the sake of completeness we shall provide the proof.

Lemma A.1 *Consider a linear operator*

$$\mathcal{G}(u) := -tr(\tau\tau^T D^2u) - \beta \cdot Du$$

where τ is a matrix whose columns verify the Hörmander condition (3.1), τ and β are smooth functions with

$$|\tau(x)|, |\beta(x)| \leq C(1 + |x|) \quad \forall x \in \mathbb{R}^N.$$

Then, conditions (2.8) and (2.9) are equivalent; namely the following properties are equivalent:

(i) *there exists $w \in C^\infty(\mathbb{R}^N)$ such that*

$$\mathcal{G}(w) \geq 1 \quad \text{in } \overline{B(0, R_0)}^C, \quad w \geq 0 \quad \text{in } \overline{B(0, R_0)}^C, \quad \lim_{|x| \rightarrow +\infty} w = +\infty$$

for some constant $R_0 > 0$;

(ii) there exists $\bar{w} \in C^\infty(\mathbb{R}^N)$ such that

$$\mathcal{G}(\bar{w}) + \chi \bar{w} = \phi \quad \text{in } \mathbb{R}^N, \quad \lim_{|x| \rightarrow +\infty} \bar{w} = +\infty$$

for some C^∞ functions χ and ϕ with $\lim_{|x| \rightarrow +\infty} \phi = +\infty$, $\chi \geq 0$ and $\text{supp} \chi$ compact.

Proof. For completeness, we report the arguments of [25]. As one can easily check, property (ii) obviously implies property (i) (possibly adding a constant).

Now, assuming (i), we want to prove (ii). We denote $K := \max_{B(0, R_0)} |w|$ and $K_{\mathcal{G}} := \max_{B(0, R_0)} |\mathcal{G}(w)|$. We fix $\chi \in C_0^\infty(\mathbb{R}^N)$ such that $\chi \geq 0$, $\chi = 1$ in $B(0, R_0)$ and $\text{supp} \chi \subset B(0, 2R_0)$. We claim that the function $w^b(x) := w(x) + K + K_{\mathcal{G}} + 1$ satisfies

$$(A.1) \quad \mathcal{G}(w^b) + \chi w^b =: f^*(x) \geq 1 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x| \rightarrow +\infty} w^b = +\infty.$$

Indeed, the latter property is an immediate consequence of (i).

Moreover, for $|x| \leq R_0$, we have

$$\mathcal{G}(w^b) + \chi w^b \geq -K_{\mathcal{G}} + \chi(w + K + K_{\mathcal{G}} + 1) \geq 1$$

while, for $|x| \geq R_0$, we have

$$\mathcal{G}(w^b) + \chi w^b \geq \mathcal{G}(w) \geq 1;$$

hence, our claim (A.1) is proved.

Let us now consider a regular partition of unity $\{\phi_i\}_{i \in \mathbb{N}}$ such that $\phi_i \geq 0$,

$$\sum_{i=1}^{\infty} \phi_i(x) = 1, \quad \text{supp } \phi_i \subset B(0, i+1) \setminus B(0, i-1),$$

$$\phi_i = 1 \text{ on } B(0, i + \frac{1}{2}) \setminus B(0, i - \frac{1}{2}).$$

We claim that there exists a regular solution to

$$(A.2) \quad \mathcal{G}(W_n) + \chi W_n = \sum_{i=1}^n \phi_i \quad \text{in } \mathbb{R}^N, \quad 0 \leq W_n \leq w^b.$$

In order to prove this existence, it is expedient to introduce, for $m \geq n+1$ and $\epsilon > 0$, the following boundary value problems

$$(A.3) \quad \begin{cases} (\mathcal{G} - \epsilon \Delta)(W_{nm}^\epsilon) + \chi W_{nm}^\epsilon = \sum_{i=1}^n \phi_i & \text{in } B(0, m) \\ W_{nm}^\epsilon = 0 & \text{on } \partial B(0, m). \end{cases}$$

By the non-degeneracy of the operator, the comparison principle applies to problems (A.3). Hence, the Perron's method ensures that there exists a unique solution to (A.3). By standard arguments in hypoelliptic theory (see [22], [29]), as $\epsilon \rightarrow 0^+$, $W_{nm}^\epsilon(x)$ converges to $W_{nm}(x)$ in $B(0, m)$, where W_{nm} is the solution to

$$(A.4) \quad \begin{cases} \mathcal{G}(W_{nm}) + \chi W_{nm} = \sum_{i=1}^n \phi_i & \text{in } B(0, m) \\ W_{nm} = 0 & \text{on } \partial B(0, m), \end{cases}$$

where the boundary condition is attained only in the viscosity sense. We observe that the Hörmander's condition guarantees the comparison principle for (A.4); since 0 and w^\flat are respectively a sub- and a supersolution, there holds true $0 \leq W_{nm} \leq w^\flat$ in $B(0, m)$. On the other hand, for $m_1 > m$, still by comparison principle, we infer $W_{nm_1}^\epsilon(x) \geq W_{nm}^\epsilon(x)$ for every $x \in B(0, m)$; so, as $\epsilon \rightarrow 0^+$, we get $W_{nm_1}(x) \geq W_{nm}(x)$ for every $x \in B(0, m)$, namely, the sequence $\{W_{nm}\}_m$ is nondecreasing and locally bounded. Passing to the limit and using the regularity theory for hypoelliptic operators (see [13]), we accomplish the proof of our claim (A.2).

By (A.2), the functions $w_i(x) := W_i(x) - W_{i-1}(x)$ solve

$$\mathcal{G}(w_i) + \chi w_i = \phi_i \quad \text{in } \mathbb{R}^N$$

and verify: $\sum_{i=1}^{\infty} w_i(x) < \infty$ in \mathbb{R}^N .

Let us recall an elementary result: for any $\sum_{i=1}^{\infty} a_i < +\infty$ with $a_i \geq 0$, there exists a sequence $\{\lambda_i\}_i$ such that $\lim_{i \rightarrow +\infty} \lambda_i = +\infty$ and $\sum_{i=1}^{\infty} \lambda_i a_i < +\infty$. Then in our case there exists a sequence $\{\lambda_i\}_i$ such that $\lim_{i \rightarrow +\infty} \lambda_i = +\infty$ and $\sum_{i=1}^{\infty} \lambda_i w_i(0) = K < +\infty$.

Let $n_0 \in \mathbf{N}$ be fixed. Let us denote by $w_n^\sharp(x) := \sum_{i=1}^n \lambda_i w_i(x)$. In $B(0, n_0)$

$w_n^\sharp(x)$ satisfies:

$$(A.5) \quad \mathcal{G}(w_n^\sharp) + \chi w_n^\sharp = \sum_{i=1}^{n_0+1} \lambda_i \phi_i \quad w_n^\sharp \geq 0$$

and by Harnack inequality there exists a constant C_{n_0} independent of n

such that

$$\begin{aligned} \sup_{B(0, \frac{n_0}{2})} w_n^\# &\leq C_{n_0} \left(\inf_{B(0, \frac{n_0}{2})} w_n^\# + \sup_{B(0, \frac{n_0}{2})} \sum_{i=1}^{n_0+1} \lambda_i \phi_i \right) \\ &\leq C_{n_0} \left(K + \sup_{B(0, \frac{n_0}{2})} \sum_{i=1}^{n_0+1} \lambda_i \phi_i \right) = C_{n_0}^* \end{aligned}$$

This implies that in any bounded set $w^\#$ is well defined, i.e. $w^\#(x) := \sum_{i=1}^{\infty} \lambda_i w_i(x) < \infty$ for every $x \in \mathbb{R}^N$. Moreover the function $w^\#$ satisfies

$$(A.6) \quad \mathcal{G}(w^\#) + \chi w^\# = \sum_{i=1}^{\infty} \lambda_i \phi_i =: \phi, \quad w^\# \geq 0$$

with $\lim_{x \rightarrow \infty} \phi(x) = +\infty$.

In conclusion, by (A.1) and (A.6), the function $\bar{w} := w^\flat + w^\#$ satisfies (ii). \square

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